International Finance

Nonlinear Exposure and FX Options

Main issues

- When is linear hedging (forwards) insufficient?
- Sources of nonlinear exposure: operating decisions create option-like payoffs
- FX option strategies: protective put, collar, risk reversal
- FX option pricing: Garman-Kohlhagen framework
- Risk information in option prices: implied volatility, the smile, and the volatility risk premium

Recap: linear exposure

In Lecture 7, we measured exposure with the regression:

$$V_T = a + b \cdot S_T + u_T$$

- b = exposure (FC units). Hedge: sell b forward.
- This works when the relationship between firm value and the exchange rate is **linear**.

But what if the relationship is nonlinear?

- Exposure may change depending on where S is

Sources of nonlinear exposure

Why might firm value respond nonlinearly to FX?

- **Pricing power thresholds:** Firm absorbs small FX moves but adjusts prices for large ones
- **Competitive thresholds:** Below a certain rate, you lose the market entirely
- **Contractual features:** Quantity adjustments, renegotiation triggers, price caps
- **Pass-through asymmetries:** Firms pass through depreciations faster than appreciations

In all these cases, the firm's **response** to FX changes is different in different states — creating kinks, curves, and option-like payoffs.

The problem with forwards

A forward locks in **both** upside and downside equally.

If exposure has a kink:

- A forward **over-hedges** on one side of the kink
- And **under-hedges** on the other

You need an instrument with an **asymmetric payoff** — one that pays off in some states but not others.

That instrument is an **option**.

The Egress export decision

Egress is a US firm that can sell 1 unit of its product:

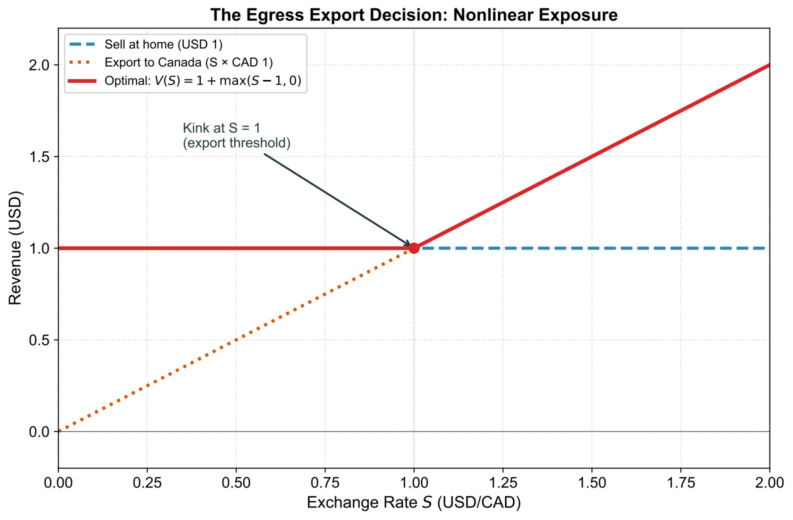
- **At home** for USD 1, or
- **In Canada** for CAD 1 (worth S in USD terms)

The firm exports only when it's profitable: when $S > 1$ (USD/CAD).

Revenue in USD:

$$V(S) = \begin{cases} 1 & \text{if } S \leq 1 \text{ (sell at home)} \\ S & \text{if } S > 1 \text{ (export)} \end{cases}$$

The Egress exposure



This exposure IS a call option

$$V(S) = 1 + \max(S - 1, 0) = 1 + \text{Call}(K = 1)$$

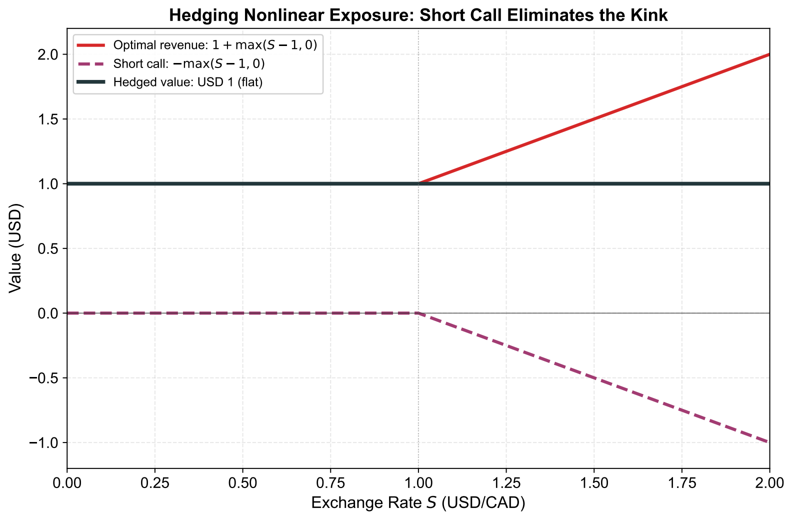
The firm's operating decision creates a **natural long call position**:

- Below the kink: exposure = 0 (sells at home, no FX sensitivity)
- Above the kink: exposure = 1 (exports, fully exposed to USD/CAD)

A forward cannot hedge this.

A forward has constant slope (delta = 1). This exposure has **changing** slope (delta = 0, then delta = 1).

Hedging with an option



General principle

When exposure is **kinked or nonlinear**, identify the option-like component:

- The kink in operating exposure corresponds to a **real business decision** (export or not, enter market or not, adjust prices or not)
- The strike price of the embedded option is the **threshold** at which the decision changes
- Hedge with the **matching option** (call or put, appropriate strike)

The operating flexibility of the firm IS an option — and option pricing theory tells us how to value and hedge it.

Smooth nonlinearity

Not all exposures have a clean kink. Some are **smoothly curved**.

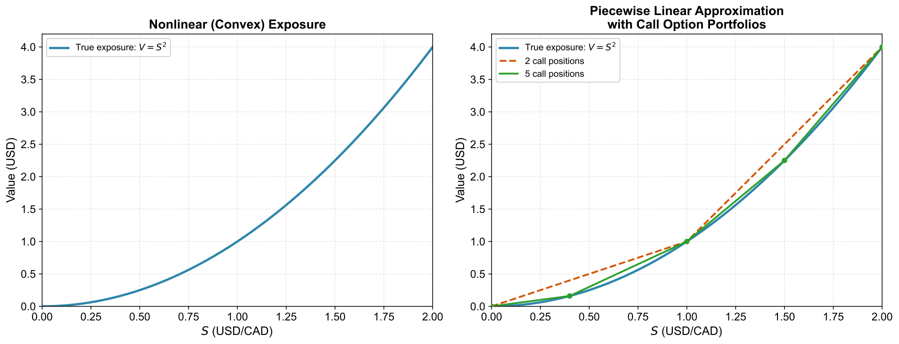
Example: Both export quantity AND price increase with S:

$$V(S) = S \times S = S^2$$

This is **convex** exposure — the slope increases as S rises.

A single forward (linear) cannot capture the curvature.

Piecewise linear approximation



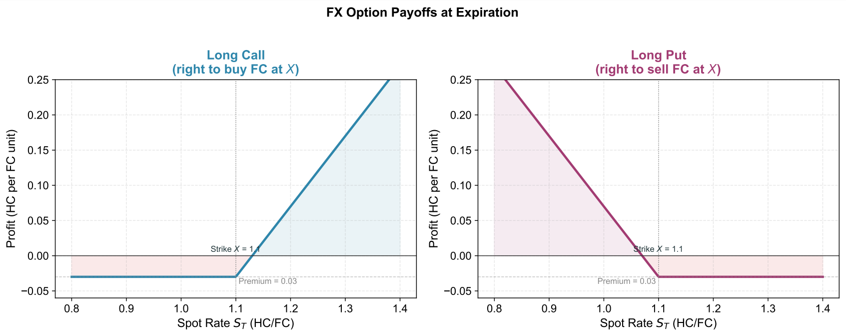
Practical implications

- Any smooth curve can be approximated by a **portfolio of calls** at different strikes
- Each call adds a kink, changing the slope — more calls means better fit

In practice:

- Most firms don't need perfect replication — **1-2 options** can capture the main nonlinearity
- Key question: **where is the kink** in your exposure? That determines the strike price.
- If exposure is approximately linear over the relevant range ⇒ use a forward (simpler, cheaper)
- If there's a threshold, cliff, or significant curvature ⇒ use options

Call and put payoffs



Call = right to buy FC at strike X. Hedges FC outflows (caps the cost).

Put = right to sell FC at strike X. Hedges FC inflows (sets a floor).

Cost: premium paid upfront (unlike forwards, which cost nothing to enter).

Strategy 1: Protective put (the floor)

Firm will **receive FC** $\Rightarrow$ buys a put to set a floor on HC value.

- If $S_T < X$: exercise put, receive X per FC unit (protected)
- If $S_T > X$: let put expire, convert at market rate (upside preserved)

Outcome: downside protected, upside preserved.

Cost: option premium (paid upfront). This is the price of asymmetric protection.

The protective put is the **simplest and most common** option strategy for corporate hedging.

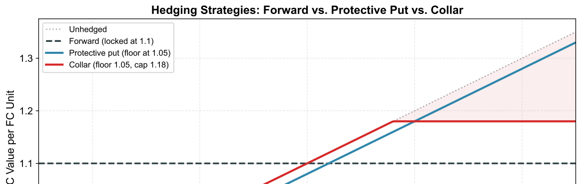
Strategy 2: Collar

Buy a put (floor) AND **sell a call** (cap) on the same FC amount.

- The call premium received offsets (part of) the put premium paid
- **Zero-cost collar:** choose strikes so premiums exactly offset

Outcome: bounded range — protected below the put strike, capped above the call strike.

Trade-off: give up upside to reduce cost.



Strategy 3: Risk reversal

Buy **OTM put** (deep downside protection) + sell **OTM call** (give up far upside).

- Near-zero premium: the call premium funds the put
- Provides **cheap tail protection** against extreme adverse moves
- Common in practice for **event risk** (elections, central bank decisions, geopolitical shocks)

Key insight: The risk reversal lets the firm insure against disaster scenarios while paying very little upfront — at the cost of giving up gains in the best-case scenarios.

When to use options vs. forwards

Use forwards when: exposure is **certain** (known amount, known date), FX view is neutral, cost minimization is the priority.

Use options when:

- Exposure is **uncertain or contingent** (tenders, bids)
- Exposure is **nonlinear** (thresholds, pricing power kinks)
- **Event risk** is high (elections, regime changes)
- Firm wants **downside protection with upside participation**

Layered hedging: forwards for near-term certain flows + options for tail risk or uncertain flows further out.

Tender hedging: the classic option application

Setup: Firm submits a bid in FC for a contract.

- If it wins: FC inflow. If it loses: nothing.

Forward hedge is dangerous: if bid rejected, firm has a naked forward position (must deliver FC it doesn't have).

Put option solves this:

- If bid accepted $\Rightarrow$ exercise put to lock in exchange rate
- If bid rejected $\Rightarrow$ let put expire (lose premium only)

This is why option markets exist for corporates — they hedge **contingent** exposures

The Garman-Kohlhagen formula

The Black-Scholes model adapted for FX (Garman-Kohlhagen, 1983):

$$c = e^{-rT}\left[F \cdot N(d_1) - X \cdot N(d_2)\right]$$

$$p = e^{-rT}\left[X \cdot N(-d_2) - F \cdot N(-d_1)\right]$$

where $d_1 = \frac{\ln(F/X)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}$

- Uses the **forward rate** F as the underlying (accounts for interest rate differential via CIP)
- Key input: **volatility** σ — the only unobservable

What drives option prices

Five inputs determine the option price:

Input	Effect on call price	Effect on put price
Forward rate $F \uparrow$	$\uparrow$	$\downarrow$
Strike $X \uparrow$	$\downarrow$	$\uparrow$
Volatility $\sigma \uparrow$	$\uparrow$	$\uparrow$
Time to expiry $T \uparrow$	$\uparrow$	$\uparrow$
HC interest rate $r \uparrow$	$\downarrow$	$\downarrow$

Volatility is the critical input: higher vol $\Rightarrow$ more expensive options (more potential payoff).

Put-call parity in FX

A call and put with the same strike and expiry are linked by arbitrage:

$$c - p = \frac{F - X}{1 + r}$$

- If you know the call price, you get the put price for free (and vice versa)
- A **forward** is equivalent to: long call + short put at strike F
- At-the-money forward ($X = F$): $c = p$ (call and put have equal value)

Put-call parity is a **no-arbitrage condition** — it holds by the same logic as CIP.

Replication and delta

- An option can be **replicated** by a dynamic portfolio of a forward + a bond
- The **hedge ratio** (delta, Δ) tells you how many forwards you need:
  - Call delta: between 0 and 1
  - Put delta: between -1 and 0

Key difference from forwards:

- A forward has **constant** delta = 1
- An option has **changing** delta — it varies with S

This connects to the nonlinear exposure discussion: the **changing delta** of an option is what allows it to match nonlinear exposure that a constant-delta forward cannot.

One-period binomial example

Setup: $S_0 = 1000$, $r = 5\%$, $r^* = 4\%$. Call with strike $X = 1050$.

- Up state: $S_u = 1100 \Rightarrow$ call pays $c_u = 50$
- Down state: $S_d = 950 \Rightarrow$ call pays $c_d = 0$
- Forward rate: $F = S_0 \times \frac{1+r}{1+r^*} = 1000 \times \frac{1.05}{1.04} \approx 1010$

Replicating portfolio: Δ forwards + B in bonds.

$$\Delta(S_u - F) + B(1+r) = c_u \quad \Rightarrow \quad \Delta(1100 - 1010) + 1.05B = 50$$
$$\Delta(S_d - F) + B(1+r) = c_d \quad \Rightarrow \quad \Delta(950 - 1010) + 1.05B = 0$$

Solving the binomial model

Step 1 — Delta (exposure):

$$\Delta = \frac{c_u - c_d}{S_u - S_d} = \frac{50 - 0}{1100 - 950} = \frac{1}{3}$$

The call's exposure is $\frac{1}{3}$ of a forward — it moves $\frac{1}{3}$ as much as the spot rate.

Step 2 — Bond position: From the down-state equation: $B = \frac{60/3}{1.05} = \frac{20}{1.05} \approx 19.05$

Step 3 — Call price: Cost of replicating portfolio (forward costs zero to enter):

$$c_0 = B = 19.05$$

Implied volatility

Given an observed option price, **invert** the GK formula to extract the market's expectation of future volatility.

This is **implied volatility** (σ_{imp}):

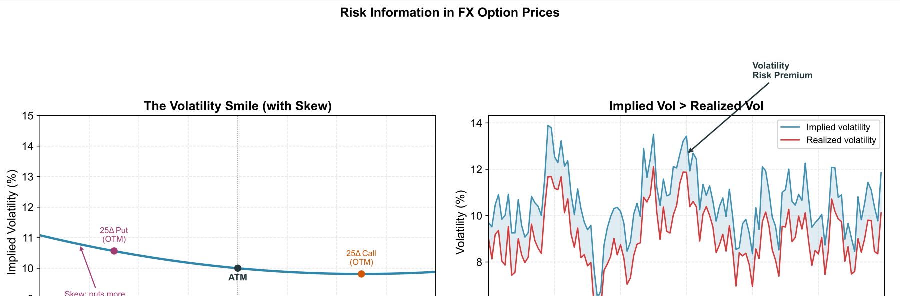
$$c_{\text{market}} = \text{GK}(F, X, T, r, \sigma_{\text{imp}})$$

- Unlike historical (realized) volatility, implied vol is **forward-looking**
- It reflects the market's best assessment of **future** FX uncertainty
- It is the single most important quantity in option markets

The volatility smile and skew

The GK model assumes constant σ across all strikes. In reality, σ varies:

- **Smile:** OTM puts and OTM calls are both more expensive than ATM $\Rightarrow$ U-shape
- **Skew:** OTM puts are more expensive than OTM calls $\Rightarrow$ the market prices **crash risk** more heavily



The volatility risk premium

On average, implied volatility **exceeds** realized volatility:

$$E[\sigma_{\text{imp}}] > E[\sigma_{\text{realized}}]$$

- Option sellers earn a premium for **bearing volatility risk**
- The wedge is the **volatility risk premium** (VRP)
- VRP = compensation for the risk that volatility spikes unexpectedly

Why it matters: When a firm buys a put to hedge, it pays the VRP on top of “fair” value. Hedging with options is more expensive than a pure probability calculation would suggest — because the seller demands compensation for risk.

Risk reversals as sentiment indicators

Risk reversal = IV of 25Δ call — IV of 25Δ put

- **Positive RR:** calls more expensive $\Rightarrow$ market expects FC appreciation
- **Negative RR:** puts more expensive $\Rightarrow$ market expects FC depreciation

Corporate treasurers and traders use risk reversals as a **real-time sentiment gauge**:

- A sharp move in the RR signals changing market expectations
- Useful for timing hedge decisions (though not for speculation!)

Risk reversals are quoted directly in the interbank FX options market — they are one of the primary instruments traded.

Connection to course themes

The volatility risk premium is **observable evidence that FX risk is priced.**

- **UIP failure** (Lecture 5): high-rate currencies don't depreciate as predicted $\Rightarrow$ carry trade earns risk premia
- **Vol risk premium** (this lecture): implied vol > realized vol $\Rightarrow$ option sellers earn risk premia

Both are **compensation for bearing risk** — the unifying theme of the course.

Every time the firm hedges (with forwards or options), it is **paying or receiving these risk premia** — whether it realizes it or not.

Summary

- **Linear exposure** $\Rightarrow$ hedge with forwards. **Nonlinear exposure** $\Rightarrow$ need options.
- Operating decisions (export thresholds, pricing power) create **option-like exposure**
- Key strategies: protective put (floor), collar (bounded range), options for contingent exposures (tenders)
- Garman-Kohlhagen prices FX options using the **forward rate** and **volatility**
- Option prices contain **risk information**: implied vol, skew, vol risk premium
- The vol risk premium connects to UIP failure — both are evidence that FX risk is priced